

Fixed point sets of equivariant fiber-preserving maps

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Abstract

Given a selfmap $f : X \rightarrow X$ on a compact connected polyhedron X , H. Schirmer gave necessary and sufficient conditions for a nonempty closed subset A to be the fixed point set of a map in the homotopy class of f . R. Brown and C. Soderlund extended Schirmer's result to the category of fiber bundles and fiber-preserving maps. The objective of this paper is to prove an equivariant analogue of Brown-Soderlund theorem result in the category of G -spaces and G -maps where G is a finite group.

1 Introduction and statement of results

A well-known and important question in classical topology is the *fixed point property*. Recall that a topological space X is said to have the fixed point property if every (continuous) map $f : X \rightarrow X$ must have a fixed point $x_0 \in X$ such that $f(x_0) = x_0$. A related question is the so-called *complete invariance property for deformation* (CIPD). We say that X has the CIPD if for any nonempty closed subset $A \subset X$, there exists a selfmap $f : X \rightarrow X$ homotopic to the identity 1_X such that $A = \text{Fix}(f) = \{x \in X \mid f(x) = x\}$. In [9], H. Schirmer generalized the concept of CIPD and gave necessary and sufficient conditions for a nonempty closed subset A to be the fixed point set of a map g in the homotopy class of a given selfmap f . That is, given a map $f : X \rightarrow X$, Schirmer determined when a closed nonempty subset A can be realized as $A = \text{Fix}(g)$ for some g homotopic to f . Upon relaxing the conditions given by Schirmer, C. Soderlund together with R. Brown [3] generalized Schirmer's result to fiber-preserving maps of fiber bundles.

Suppose that X is a compact connected polyhedron without local cutpoints and A is a closed subset imbedded inside a subpolyhedron K that can be *by-passed in X* , that is, every path C in X with $C(0), C(1) \in X - K$, is homotopic to a path C' in $X - K$ relative to the endpoints. H. Schirmer [9] introduced the following two conditions to realize A as the fixed point set of a selfmap in the homotopy class $[f]$.

- (C1) if there exists a homotopy $H_A : A \times [0, 1] \rightarrow X$ from $f|_A$ to the inclusion $i : A \hookrightarrow X$;
- (C2) if for every essential fixed point class \mathbb{F} of f , there exists a path $\alpha : [0, 1] \rightarrow X$ with $\alpha(0) \in \mathbb{F}, \alpha(1) \in A$ and $\{\alpha(t)\} \sim \{f \circ \alpha(t)\} * \{H_A(\alpha(1), t)\}$ relative to the endpoints.

Soderlund [10, Theorem 3.5] showed, by relaxing the assumption on A given by Schirmer, the following result.

Theorem 1.1. Let X be a compact, connected polyhedron with no local cut points and A be a closed locally contractible subspace of X such that $X - A$ is not a 2-manifold and A can be by-passed in X . Then $A = \text{Fix}(g)$ for some $g \sim f$ if and only if (C1) and (C2) are satisfied.

Subsequently, R. Brown and C. Soderlund [3] introduced analogous conditions in the fiber-preserving setting. Let $\mathfrak{F} = (E, p, B; Y)$ be a (locally trivial) fiber bundle and $f : E \rightarrow E$ a fiber preserving map.

- (C1 $_{\mathfrak{F}}$) if there exists a fiber preserving homotopy $H_A : A \times [0, 1] \rightarrow E$ from $f|_A$ to the inclusion $i : A \hookrightarrow E$;
- (C2 $_{\mathfrak{F}}$) if for every essential fixed point class \mathbb{F} of f , there exists a path $\alpha : [0, 1] \rightarrow E$ with $\alpha(0) \in \mathbb{F}, \alpha(1) \in A$ and $\{\alpha(t)\} \sim \{f \circ \alpha(t)\} * \{H_A(\alpha(1), t)\}$ relative to the endpoints.

Following the terminology of [3], we call (X, A) a *suitable pair* if X is a finite polyhedron with no local cut points and A is a closed locally contractible subspace of X such that $X - A$ is not a 2-manifold and A can be by-passed in X .

In [3], it was shown that conditions (C1 $_{\mathfrak{F}}$) and (C2 $_{\mathfrak{F}}$) are also sufficient. The following is their main result.

Theorem 1.2. Let $\mathfrak{F} = (E, p, B; Y)$ be a fiber bundle where E, B and Y are connected finite polyhedra, $f : E \rightarrow E$ a fiber preserving map and A a closed locally contractible sub-bundle of E such that each component $p(A)_j$ of $p(A)$ is contractible and $(B, p(A)), (Y, Y_j)$ for all sub-bundle fibers Y_j of A , are suitable pairs. Suppose (C1 $_{\mathfrak{F}}$) and (C2 $_{\mathfrak{F}}$) are satisfied and A intersects every essential fixed point class of $f_{b_j} : p^{-1}(b_j) \rightarrow p^{-1}(b_j)$ for at least one b_j in each component $p(A)_j$. If Z is a closed bundle subset of A that intersects every component of A , then there exists a map $g : E \rightarrow E$ that is fiber preserving and fiberwise homotopic to f ($g \sim_{\mathfrak{F}} f$) such that $\text{Fix}(g) = Z$.

In particular, when $Z = A$, this theorem shows that $(C1_{\mathfrak{F}})$ and $(C2_{\mathfrak{F}})$ are necessary and sufficient for $A = \text{Fix}(g)$ for some $g \sim_{\mathfrak{F}} f$.

Many applications involve symmetries in the presence of a group action. As a result, equivariant topology has been proven to be useful in the study of nonlinear problems. In the equivariant setting, we are concerned with a group G acting on a space X together with a G -map $f : X \rightarrow X$ which respects the group action, that is, for all $\alpha \in G$, $f(\alpha x) = \alpha f(x)$ for all $x \in X$. In this case, the fixed point set $\text{Fix}(f)$ is *a priori* a G -invariant subset of X .

In [9], Schirmer observed that for a given selfmap $f : \mathbb{S}^n \rightarrow \mathbb{S}^n$ of an n -sphere, $n \geq 2$, *any* closed nonempty proper subset A of \mathbb{S}^n can be realized as the fixed point set of a map $g \in [f]$ with $\text{Fix}(g) = A$. However, such phenomenon does not hold if we impose a group action as we show in the following example, which gives the underlying motivation for this paper.

Example 1.3. Let $G = \mathbb{Z}_2$, $X = \mathbb{S}^2$ and the action is given by $\xi(x, y, z) \mapsto (-x, -y, z)$. If $A = \{(x, y, 0) \in \mathbb{S}^2\}$ then A is \mathbb{Z}_2 -invariant, but there is no \mathbb{Z}_2 -map $h : \mathbb{S}^2 \rightarrow \mathbb{S}^2$ that is \mathbb{Z}_2 -homotopic to the identity map $Id : \mathbb{S}^2 \rightarrow \mathbb{S}^2$ such that $\text{Fix}(h) = A$.

In fact, suppose there is a \mathbb{Z}_2 -homotopy H from Id to $h : \mathbb{S}^2 \rightarrow \mathbb{S}^2$ such that $\text{Fix}(h) = A$. Then, h preserves $X^G = \{N, S\}$, where $N = (0, 0, 1)$ and $S = (0, 0, -1)$. Hence, $h(N) = S$ and $h(S) = N$ and the path $p : I \rightarrow X^G$ defined by $p(t) = H^G(N, t)$ is such that $p(0) = Id(N) = N$ and $p(1) = h(N) = S$. But, this is impossible.

In this situation, the location of A in X is more important than its topology, because if we replace A by $A' = \{(x, 0, z) \in \mathbb{S}^2\}$ then:

$$H(t, \cos \theta \sin \psi, \sin \theta \sin \psi, \cos \psi) = (\cos(\theta + t\epsilon \sin \theta) \sin \psi, \sin(\theta + t\epsilon \sin \theta) \sin \psi, \cos \psi),$$

is a \mathbb{Z}_2 -homotopy (with polar coordinates) between the identity and the \mathbb{Z}_2 -map h such that $\text{Fix}(h) = A'$.

Example 1.4. It is easy to see, by modifying the last example, that the equivariant analogue of Schirmer's result does not hold in general. Let $G = \mathbb{Z}_2$, $X = \mathbb{S}^2 \times \mathbb{S}^2$ and the action is given by $\xi((x, y, z), (x', y', z')) \mapsto ((-x, -y, z), (1, 0, 0))$. The set $A = \{((x, y, 0), (1, 0, 0)) \in X\}$ is \mathbb{Z}_2 -invariant and $X^G = \{N, S\} \times \{(1, 0, 0)\}$ consists of two points. The same argument as in Example 1.3 shows that A cannot be the fixed point set of any map \mathbb{Z}_2 -homotopic to the identity map while (X, A) satisfies the conditions of Schirmer's result for A can be by-passed since A has codimension 3 in X .

The main objective of this paper is to give an equivariant analogue of Schirmer's result and of Brown-Soderlund's result. This paper is organized as follows. In the first section, we briefly recall the non-equivariant results of [9] and [3] and review some basic background on G -maps and G -spaces where G denotes a compact Lie group. Then we review the necessary equivariant Nielsen fixed point theory from [13]. In section 2, we prove our first main result, an equivariant analogue of [9]:

Theorem 1.5. Let G be a compact Lie group, X be a compact and smooth G -manifold and A be a nonempty, closed, locally contractible G -subset of X such that for each finite WK we assume that $\dim(X^K) \geq 3$, $\dim(X^K) - \dim(X^K - X_K) \geq 2$ and A^K is by-passed in X^K , for all $(K) \in \text{Iso}(X)$. Suppose that the following conditions holds for a G -map $f : X \rightarrow X$:

- (C_G1) there exists a G -homotopy $H_A : A \times I \rightarrow X$ from $f|_A$ to the inclusion $i : A \hookrightarrow X$;
- (C_G2) for each finite WK , for every WK -essential fixed point class F of $f^K : X^K \rightarrow X^K$ there exists a path $\alpha : I \rightarrow X^K$ with $\alpha(0) \in F$, $\alpha(1) \in A^K$, and $\{\alpha(t)\} \sim \{f^K \circ \alpha(t)\} * \{H_A^K(\alpha(1), t)\}$.

Then for every closed G -subset Φ of A that has nonempty intersection with every component of A there exists a G -map $h : X \rightarrow X$, G -homotopic to f with $\text{Fix}(h) = \Phi$.

In the last section, we apply Theorem 1.5 to prove an equivariant analogue of [3] when G is finite:

Theorem 1.6. Let G be a finite group, $\mathfrak{F} = (X, p, B, Y)$ be a G -fiber bundle where X , B and Y are compact and smooth G -manifolds, $\dim(B^K) \geq 3$, $\dim(B^K) - \dim(B^K - B_K) \geq 2$, for all $(K) \in \text{Iso}(B)$, $\dim(Y^K) \geq 3$, $\dim(Y^K) - \dim(Y^K - Y_K) \geq 2$, for all $(K) \in \text{Iso}(Y)$.

Let A be a nonempty, closed, locally contractible G -subset of X such that (X, A) is G -fiber bundle pair with respect to the fiber bundle \mathfrak{F} , $p(A)$ be a closed G -subset of B such that each component $p(A)_j$ of $p(A)$ is equivariantly contractible and $p^K(A^K)$ is by-passed in B^K , for all $(K) \in \text{Iso}(B)$. Let Y_j be a subbundle fiber of A such that Y_j is a closed and locally contractible G -subset of Y and Y_j^K is by-passed in Y^K , for all $(K) \in \text{Iso}(Y)$, and $f : X \rightarrow X$ be a G -fiber-preserving map such that A^K intersects every essential WK -fixed point class of $f_{b_j}^K : WK(p^K)^{-1}(\{b_j\}) \rightarrow WK(p^K)^{-1}(\{b_j\})$ for at least one b_j in each component $p^K(A^K)_j$, for all $(K) \in \text{Iso}(X)$. Suppose that the following conditions hold for f and A :

- (C_G1) $_{\mathfrak{F}}$ there exists a G -fiberwise-homotopy $H_A : A \times I \rightarrow X$ from $f|_A$ to the inclusion $i : A \hookrightarrow X$;
- (C_G2) $_{\mathfrak{F}}$ for every WK -essential fixed point class F of $f^K : X^K \rightarrow X^K$ there exists a path $\alpha : I \rightarrow X^K$ with $\alpha(0) \in F$, $\alpha(1) \in A^K$, and $\{\alpha(t)\} \sim \{f^K \circ \alpha(t)\} * \{H_A^K(\alpha(1), t)\}$.

Then for every nonempty closed G -bundle subset Φ of A that intersects every component of A there exists a G -fiber-preserving map h , G -fiberwise homotopic to f with $\text{Fix}(h) = \Phi$.

In order to establish the notations, let G be a topological group and X be a (left) G -space. Given a subgroup K of G we denote by NK the normalizer of K in G , $WK = \frac{NK}{K}$ is the Weyl group of K in G . The orbit type of K is the conjugacy class of K in G denoted by (K) . If (K_1) is subconjugate to (K_2) , we write $(K_1) \leq (K_2)$.

If $x \in X$, then $G_x = \{g \in G; gx = x\}$ denotes the isotropy subgroup of $x \in X$, and (G_x) is called an isotropy type of X . We denote by $\text{Iso}(X)$ the set of isotropy types of X . Moreover, $X^K = \{x \in X; K \leq G_x\}$, $X^{(K)} = \{x \in X; (K) \leq (G_x)\}$, $X_K = \{x \in X; G_x = K\}$ and $X_{(K)} = \{x \in X; G_x \subset (K)\}$.

If $\text{Iso}(X)$ is finite (in particular when G is finite), we can choose an admissible ordering on $\text{Iso}(X)$ such that $(K_i) \leq (K_j)$ implies $i \leq j$. Then we have a filtration of G -subspaces $X_1 \subset \cdots \subset X_n = X$ where $X_i = \{x \in X; (G_x) = (H_j) \text{ for some } j \leq i\}$

If $f : X \rightarrow X$ is a G -map, then $f^K = f|_{X^K} : X^K \rightarrow X^K$ is a WK -map. Let $\mathcal{F} = \{(K) \in \text{Iso}(X) \mid |WK| < \infty\}$ and $(K) \in \mathcal{F}$. If $x, y \in \text{Fix}(f^K)$ then $x \sim_K y$ if either $y = \alpha x$ for some $\alpha \in WK$ or $\exists \sigma : [0, 1] \rightarrow X^K$ such that $\sigma \sim f^K \circ \sigma$ relative to endpoints. Then \sim_K is an equivalence relation on $\text{Fix}(f^K)$ and the equivalence classes are called the WK -fixed point classes of f^K . Evidently, a WK -fixed point class \mathcal{W} is a disjoint union of a finite number of ordinary fixed point classes W_1, \dots, W_r of f^K and thus the fixed point index $\text{ind}(\mathcal{W})$ is defined as $\text{ind}(\mathcal{W}) = \sum_i \text{ind}(W_i)$. A WK -fpc (fixed point class) \mathcal{W} is *essential* if $\text{ind}(\mathcal{W}) \neq 0$. For further information on equivariant Nielsen fixed point theory, see [13]. Throughout, by a smooth G -manifold X , we assume that the fixed point set X^H is a smooth connected submanifold for each isotropy subgroup $H \leq G$.

2 Proof of Theorem 1.5 - An equivariant analogue of a result of Soderlund-Schirmer

If X is a smooth G -manifold and A is a closed smooth G -submanifold of X , G being a finite group, then there exists a smooth equivariant triangulation $f_1 : (K, K_0) \rightarrow (X, A)$ as proved in [8]. If B is another closed smooth G -submanifold of X then there is a smooth equivariant triangulation $f_2 : (L, L_0) \rightarrow (X, B)$ and G -subdivisions K' of K and L' of L such that $f_1'^{-1} \circ f_2' : |L'| \rightarrow |K'|$ is a simplicial G -homeomorphism, where f_1' and f_2' are smooth G -triangulations (see [8]).

By Corollary 3.3.5 of [11] and G being finite, we can find unique G -subcomplexes L'_0 of L and K'_0 of K such that L'_0 is a refinement of L_0 and K'_0 is a refinement of K_0 . Then, $f_1'^{-1} \circ f_2'(L'_0) = K_1$ is a G -subcomplex of K' and a G -triangulation of B . In fact, $f_1'(K_1) = f_1' \circ f_1'^{-1} \circ f_2'(L'_0) = f_2'(L'_0) = B$. Hence, by induction if $\{A_i\}_{i=1}^n$ is a finite collection of closed smooth G -submanifolds of X then there exists a smooth equivariant triangulation $f : K \rightarrow X$ and a finite collection of G -subcomplexes $\{L_i\}_{i=1}^n$ of K such that L_i is a G -triangulation of A_i , for $i = 1, \dots, n$.

To realize A as the fixed point set of some $h : X \rightarrow X$, it is necessary to remove every fixed point $x \in X$ of $f : X \rightarrow X$ outside of A . Hence, we need to extend the notion of *neighborhood by-passed* for a closed subset A as in [10, Definition 2.1] in order to handle these undesired fixed points.. Thus, a G -invariant subset A is said to be *G -neighborhood by-passed* if there

exists an invariant open subset $U \subset X$ such that $A \subset U$ and U can be by-passed in X .

We observe that if (X, A) is a G -ENR pair then A is an invariant neighborhood retract in X and if $\{Y_i\}_{i=1}^n$ is a finite collection of closed smooth G -submanifolds such that $\dim(Y_i) + 1 < \dim(X)$, then $A \cup Y$ ($Y = \bigcup_{i=1}^n Y_i$) remains a by-passed G -subset of X provided A is by-passed in X . Furthermore, a close inspection of the proof of Theorem 2.2 of [10] indicates that the same argument works for the same result in the equivariant setting. That is, if A is a by-passed locally contractible G -subset of X then A is G -neighborhood by-passed, for X a compact smooth G -manifold with $\dim(X) \geq 3$. To see that, we note that if K is the G -triangulation of X then there is a by-passed neighborhood (may not be equivariant) U of A in $|K|$. We obtain the open G -subpolyhedron:

$$St(A, K) = \bigcup_{\substack{|t| \cap A \neq \emptyset \\ t \in K}} |t|,$$

such that $\overline{St(A, K)}$ is a subset of U by taking a G -refinement K' of K if necessary, where t is a simplex of K . Therefore, if $p : I \rightarrow X$ is a path with endpoints in $U - \overline{St(A, X)}$ and outside $St(A, X')$ then using Corollary 3.3.11 of [11] we deform p out of $St(A, X')$.

Thus, if $\{Y_i\}_{i=1}^n$ is a finite collection of closed smooth G -submanifolds such that $\dim(Y_i) + 1 < \dim(X)$ (thus each Y_i has codimension at least 2 in X so that Y_i can be by-passed in X), then $A \cup Y$ ($Y = \bigcup_{i=1}^n Y_i$) remains a by-passed G -subset of X using a finite collection of G -subcomplex $\{L_i\}_{i=1}^n$ of K such that L_i is a G -triangulation of Y_i , for $i = 1, \dots, n$.

The next lemma shows how the fixed points outside A may be removed (see also [7]).

Lemma 2.1. Let $\{Y_i\}_{i=1}^n$ be a finite collection of closed G -submanifolds of the G -manifold X such that $\dim(Y_i) + 1 < \dim(X)$ and the action of G outside $Y = \bigcup_{i=1}^n Y_i$ is free, where G is a finite group. Let $f : X \rightarrow X$ be a G -selfmap, A be a non-empty closed locally contractible and by-passed G -subset of X such that $A \subset \text{Fix}(f)$, there are no fixed points of f in $Y - A$, and f has a finite number of fixed points in $X - (A \cup Y)$. Let x_0 and x_1 be two fixed points of f that are G -Nielsen equivalent from different orbits such that $x_0 \in X - (A \cup Y)$ and $x_1 \in X - (A \cup Y)$ or $x_1 \in \partial(A)$, where $\partial(A)$ is the boundary of A in X and $q : I \rightarrow X$ a path with end points $q(0) = x_0$ and $q(1) = x_1$ such that $f \circ q$ is homotopic to q relative to the endpoints.

Then, f is G -homotopic, relative to $(A \cup Y)$, to a G -selfmap $h : X \rightarrow X$ such that $\text{Fix}(h) = \text{Fix}(f) - G\{x_0\}$.

Proof of Lemma 2.1: Since A is locally contractible and can be by-passed in X , the discussion above shows that A is G -neighborhood by-passed in X . Furthermore, $A \cup Y$ can be by-passed in X . Thus, the path q is homotopic, relative to endpoints, to a path $q'(t)$ such that for $0 \leq t < 1$, $q'(t) \in X - (A \cup Y)$ with $q'(0) = x_0, q'(1) = x_1$. Since G acts freely on $X - Y$ and hence on $X - (A \cup Y)$, taking the G -translates of q' yields $|G|$ paths from

the orbit $G\{x_0\}$ to the orbit $G\{x_1\}$. Note that the segments $G\{q'([0, 1])\}$ are disjoint while $\{G\{q'(1)\}\}$ consists of $[G : G_{x_1}]$ distinct endpoints. Here, the isotropy subgroup G_{x_1} at x_1 is trivial if $x_1 \in X - (A \cup Y)$. Now we coalesce these two fixed orbits in the same fashion as in [14, Lemma 3.1]. (For slightly more general spaces in which normal arcs are used, see [7, Theorem 2].)

□

We will prove Theorem 2.2 before Theorem 1.5 and for the same reason we prove Theorem 2.2 by first establishing Lemma 2.3 and Lemma 2.5.

Theorem 2.2. Let G be a compact Lie group, X be a compact smooth G -manifold and A be a nonempty, closed, locally contractible G -subset of X such that for each finite WK we assume that $\dim(X^K) \geq 3$, $\dim(X^K) - \dim(X^K - X_K) \geq 2$ and A^K is by-passed in X^K , for all $(K) \in \text{Iso}(X)$. Then, given a G -map $f : X \rightarrow X$ there exists a G -map $h : X \rightarrow X$ G -homotopic to f with $\text{Fix}(h) = A$ if, and only if, the conditions (C_G1) and (C_G2) , given in Theorem 1.5, hold for f relative to A .

Lemma 2.3. Let G be a compact Lie group, X be a G -space G -ANR and A be a nonempty closed G -subset of X . If $f : X \rightarrow X$ is a G -map G -homotopic to $h : X \rightarrow X$ such that $\text{Fix}(h) = A$ then the conditions (C_G1) and (C_G2) given by Theorem 1.5 hold for f relative to A .

Proof of Lemma 2.3: Let $H : X \times I \rightarrow X$ be a G -homotopy which starts at f and ends at h . Then $\overline{H} = H|_{(X \times \{0\}) \cup (A \times I)} : (X \times \{0\}) \cup (A \times I) \rightarrow X$ satisfies (C_G1) . If F is a WK -essential fixed point class of f^K , then, there exists a path $p : I \rightarrow X^K$ such that $p(0) \in F$ and $p(1) \in J$, where $J \subset A^K$ is a WK -essential fixed point class of h^K , H^K -related to F and $\{p(t)\} \sim \{\overline{H}^K(p(t), t)\}$. In fact,

$$\{\overline{H}^K(p(t), t)\} \sim \underbrace{\{\overline{H}^K(p(t), 0)\}}_{=\{f \circ p(t)\}} * \{\overline{H}^K(p(1), t)\}.$$

So, (C_G2) is satisfied.

□

Lemma 2.3 shows that the conditions (C_G1) and (C_G2) are necessary for $A = \text{Fix}(h)$. The example below shows that these two conditions are independent of each other.

Example 2.4. Let $G = \mathbb{Z}_2$, $X = \mathbb{S}^2$ and the action given by $\xi(x, y, z) \mapsto (-x, -y, z)$. Then, there is no \mathbb{Z}_2 -homotopy H from the identity Id to h such that $\text{Fix}(h) = \{(x, y, 0) \in \mathbb{S}^2\}$. Note that (C_G1) occurs, because the map is the identity, but (C_G2) does not. On the other hand, let $G = \mathbb{Z}_2$, $X = \mathbb{S}^3$ and the action given by $\xi(x, y, z, w) \mapsto (x, y, z, -w)$. Then, there is no \mathbb{Z}_2 -homotopy H from the antipodal map $-Id$ to h such that $\text{Fix}(h) = \{(x, y, z, 0) \in \mathbb{S}^3\}$. This time (C_G2) holds because the map is fixed point free but (C_G1) does not hold.

Lemma 2.5. Let G be a compact Lie group, X be a compact smooth G -manifold and A be a nonempty, closed, locally contractible G -subset of X such that for each finite WK we assume that $\dim(X^K) \geq 3$, $\dim(X^K) - \dim(X^K - X_K) \geq 2$ and A^K is by-passed in X^K , for all $(K) \in \text{Iso}(X)$. If the conditions (C_G1) and (C_G2) , given in Theorem 1.5, hold for a G -map $f : X \rightarrow X$ relative to A , then there exists a G -map $h : X \rightarrow X$, G -homotopic to f with $\text{Fix}(h) = A$.

Proof of Lemma 2.5: This proof follows the steps of the proof of Theorem 3.2 of [9]. Consider a G -map $\overline{H} : (X \times \{0\}) \cup (A \times I) \rightarrow X$ given by (C_G1) . It is possible to extend \overline{H} to a G -homotopy ${}_1\overline{H}_1 : \left((X \times \{0\}) \cup \left((A \cup X_1) \times I \right) \right) \rightarrow X$. As commented above, there is a closed G -invariant neighborhood V of A_1 inside X_1 and V retracts onto A_1 equivariantly. Note that WK_1 acts freely on $X_1^{K_1} = X_{K_1}$ and ${}_1h_1^{K_1}$ is a WK_1 -map. Hence, if WK_1 has positive dimension we apply Lemma 3.3 of [12] and Lemma 2.1 of [6] to extend ${}_1\overline{H}_1$ to a G -homotopy $\overline{H}_1 : (X \times \{0\}) \cup ((A \cup X_1) \times I) \rightarrow X$, relative to V . Moreover, h_1 has no fixed points in $X_1 - A_1$ and $\text{Fix}(h_1) = A$, where $h_1 = \overline{H}_1(\bullet, 1) : A \cup X_1 \rightarrow X$.

On the other hand, if WK_1 is a finite group then X^{K_1} is a WK_1 -polyhedron such that $A_1^{K_1}$ is a WK_1 -subpolyhedron and $St(A_1, X^{K_1})$ is neighborhood by-passed in X^{K_1} . We apply Lemma 3.1 of [12] and Lemma 2.1 to obtain a WK_1 -homotopy $H : (A_1 \cup X_1)^{K_1} \times I \rightarrow X^{K_1}$ which can be extended by Lemma 2.1 of [6] to a G -homotopy $\overline{H}_1 : (X \times \{0\}) \cup ((A \cup X_1) \times I) \rightarrow X$, relative to V , such that h_1 has no fixed points in $X_1 - A_1$ and $\text{Fix}(h_1) = A$, where $h_1 = \overline{H}_1(\bullet, 1) : A \cup X_1 \rightarrow X$.

By induction, we may assume that we have a G -map $\overline{H}_{i-1} : (X \times \{0\}) \cup ((A \cup X_{i-1}) \times I) \rightarrow X$ such that $\text{Fix}(h_{i-1}) = A$, where $h_{i-1} = \overline{H}_{i-1}(\bullet, 1) : A \cup X_{i-1} \rightarrow X$ and the proof follows the steps we did for WK_1 .

□

Now Theorem 2.2 follows easily from Lemma 2.3 and Lemma 2.5.

Proof of Theorem 1.5: First of all, by Theorem 2.2, there is a G -map $h_1 : X \rightarrow X$ G -homotopic to f such that $\text{Fix}(h_1) = A$. We may apply Proposition 2.5 of [12] and Theorem 4.3 of [13] to conclude that h_1 is G -homotopic to h_2 such that $h_2|_{X^K}$ has a finite number of fixed points, all of which inside $St(A^K)$ and lying in the interior of a maximal simplex of X^K and h_2 is a G -proximity map in $St(A)$ (for some G -triangulation of X).

Since Φ has nonempty intersection with every component of A we can pull the fixed points of h_2 to Φ . Let α be the G -map of Lemma VIII.C.1 of [2] and \overline{d} the equivariant bounded distance in X then we define

$$\overline{H}_3 : (X \times \{0\}) \cup (St(A) \times I) \rightarrow X$$

given by:

$$(x, t) \mapsto \begin{cases} \alpha(x, h_2(x), 1 - (1 - \bar{d}(x, \Phi))t) & \text{if } (x, t) \in St(A) \times I; \\ h_2(x) & \text{if } t = 0. \end{cases}$$

Then, we extend \bar{H}_3 , relative to $\partial(St(A))$, to a G -map $\bar{H}_4 : X \times I \rightarrow X$. By Lemma 3.1 of [12], we eliminate the fixed points of $\bar{H}_4(\bullet, 1)$ inside $X - St(A) \times \{1\}$. This finite set of fixed points can be removed because these fixed points lie in some non essential fixed point classes of $\bar{H}_4(\bullet, 1)$ since $h_2|_{X - Int(A)}$ is fixed point free. Thus, the resulting G -map is a G -homotopy $H : X \times I \rightarrow X$ connecting f to a G -map h such that $\text{Fix}(h) = \Phi$.

3 Proof of Theorem 1.6 - An equivariant analogue of a theorem of Brown-Soderlund

Throughout this last section, G will denote a finite group. Given a G -fiber-preserving map $f : X \rightarrow X$ of the total space X of a G -fiber bundle $\mathfrak{F} = (X, p, B, Y)$, it is known that the fixed point set of f is related with the fixed point set of the induced map $\bar{f} : B \rightarrow B$. However, there are equivariant homotopies that are not fiber-preserving as in the example below:

Example 3.1. Let $G = \mathbb{Z}_2$ and $X = \mathbb{S}^2 \times \mathbb{S}^1$ and the action is given by $\xi((a, b, c), \cos x + i \sin x) \mapsto ((a, b, c), \cos x - i \sin x)$. The G -map f , defined on X by setting $f((a, b, c), \cos x + i \sin x) = ((-a, -b, -c), \cos x + i \sin x)$, is the start of the following equivariant homotopy:

$$H\left(\left(\cos \theta \sin \psi, \sin \theta \sin \psi, \cos \psi\right), \cos x + i \sin x, t\right) = \left(\left(-\cos(\theta + t|\sin x|\pi) \sin \psi, -\sin(\theta + t|\sin x|\pi) \sin \psi, -\cos \psi\right), \cos x + i \sin x\right).$$

Then, $A = \{(a, b, 0) \in \mathbb{S}^2\} \times \{-i, i\} = \mathbb{S}^1 \times \{i, -i\}$ is the fixed point set of $h \in [f]_G$ where $h = H(\bullet, 1)$. Let $p = \pi_1 : \mathbb{S}^2 \times \mathbb{S}^1 \rightarrow \mathbb{S}^2$ be the projection, then $(\mathbb{S}^2 \times \mathbb{S}^1, \pi_1, \mathbb{S}^2)$ is a \mathbb{Z}_2 -fiber bundle, f is a fiber-preserving map and the induced map $\bar{f} = a : \mathbb{S}^2 \rightarrow \mathbb{S}^2$ is the antipodal map. However, $p((x, y, z), 1) = (x, y, z) = p((x, y, z), i)$ and $p \circ h((x, y, z), 1) = (-x, -y, -z)$ is different from $p \circ h((x, y, z), i) = (x, y, -z)$. So, h is not a fiber preserving map and H is not a fiber-preserving homotopy. In fact, A *cannot* be realized as the fixed point set of *any* map equivariantly fiberwise homotopic to f . To see that, we note that $X^G = \mathbb{S}^2 \times \{\pm 1\} = \mathbb{S}_1^2 \sqcup \mathbb{S}_{-1}^2$, where $(w, \pm 1) \in \mathbb{S}_{\pm 1}^2$, consists of two disjoint 2-spheres \mathbb{S}^2 . If F_t is a \mathbb{Z}_2 fiber-preserving homotopy such that $F_0 = f$ and $\text{Fix}(F_1) = A$, then F_t^G is a homotopy on X^G . Now, $f^G = F_0^G$ maps \mathbb{S}_1^2 to \mathbb{S}_1^2 and \mathbb{S}_{-1}^2 to \mathbb{S}_{-1}^2 . On the other hand, F_1 is fiber-preserving and A is the fixed point of F_1 , it follows that the induced map \bar{F}_1 fixes the circle $\{(a, b, 0) \in \mathbb{S}^2\}$ pointwise. This implies that F_1 maps the (non-fixed) point $((a, b, 0), 1)$ to the point $((a, b, 0), -1)$ so that F_1 maps the equator of \mathbb{S}_1^2 to that of \mathbb{S}_{-1}^2 , and vice versa. Thus F_1^G maps X^G to itself by

interchangeing the two disjoint spheres $\mathbb{S}_{\pm 1}^2$. The images of X^G under F_0^G and F_1^G contradict the continuity of F_t^G . Hence such an equivariant fiber-preserving homotopy F_t cannot exist.

The example above indicates the importance of modifying the conditions (C_G1) and (C_G2) and replacing them by $(C_G1)_{\mathfrak{F}}$ and $(C_G2)_{\mathfrak{F}}$ for the fiber-preserving map setting.

Lemma 3.2. Let $f : X \rightarrow X$ be a G -fiber preserving map in the total space of the G -fiber bundle $\mathfrak{F} = (X, p, B, Y)$, where X , B and Y are G -spaces ANR . Suppose that there is a G -fiber preserving homotopy connecting a G -fiber preserving map $h : X \rightarrow X$ to f such that $\text{Fix}(h) = A$ for a nonempty and closed G -subset A of X . Then the conditions $(C_G1)_{\mathfrak{F}}$ and $(C_G2)_{\mathfrak{F}}$ given in Theorem 1.6 hold for f and A .

The proof of Lemma 3.2 follows the steps of Lemma 2.3. Since $\mathfrak{F} = (X, p, B, Y)$ is a G -fiber bundle where X , B and Y are compact smooth G -manifolds, we observe that (X, p, B) is a G -fibration and there is a G -lift map $\Lambda : \Omega_p \rightarrow E^I$ such that $\Lambda(e, \alpha)(0) = e$, $p \circ \Lambda(e, \alpha)(t) = \alpha(t)$ and $\Lambda(e, p(e))(t) = e$, for all $t \in I$, where $E^I = \{\alpha : I \rightarrow E; \alpha \text{ is a path}\}$ and $\Omega_p = \{(e, \alpha) \in X \times B^I; p(e) = \alpha(0)\}$.

Remark 3.3. We should point out that Lemma 3.2 holds for any compact Lie group G if we modify condition $(C_G2)_{\mathfrak{F}}$ by only considering those (K) 's with $|WK| < \infty$.

The next proposition is an equivariant analogue of Theorem 2.1 of [1].

Proposition 3.4. Let $\overline{H} : (X \times \{0\}) \cup (A \times I) \rightarrow E$ be a G -map in the G -fibration $\mathfrak{F} = (E, p, B)$, where E is a G -ANR, A is a closed G -subset of X , (X, A) is a G -metric pair and $p \circ \overline{H}(x, 0) = p \circ \overline{H}(x, t)$ for all $(x, t) \in A \times I$. Then \overline{H} can be extended to a G -homotopy $H : X \times I \rightarrow E$ such that $p \circ H(x, 0) = p \circ H(x, t)$ for all $(x, t) \in X \times I$.

Proof of Proposition 3.4: Let $H' : X \times I \rightarrow E$ a G -extension of \overline{H} . Then H' is given by:

$$\begin{aligned} H' : X &\rightarrow E^I \\ x &\mapsto H'(x, \bullet) : I \rightarrow E \\ t &\mapsto H'(x, t). \end{aligned}$$

Then define $H(x, t) = \Lambda(H'(x, t), p(H'(x, \bullet))_t)(1)$, where $p(H'(x, \bullet))_t(s) = p(H'(x, (1-s)t))$ and Λ is a G -lift map.

□

Lemma 3.5. Let $\mathfrak{F} = (X, p, B, Y)$ be a G -fiber bundle where X , B and Y are compact and smooth G -manifolds, $\dim(B^K) \geq 3$, $\dim(B^K) - \dim(B^K - B_K) \geq 2$, for all $(K) \in \text{Iso}(B)$, A be a nonempty, closed, locally contractible G -subset of X such that $p(A)$ be a closed G -subset of B and $p^K(A^K)$ is by-passed in B^K , for all $(K) \in \text{Iso}(B)$, and $f : X \rightarrow X$ a G -

fiber preserving map such that conditions $(C_G1)_{\mathfrak{F}}$ and $(C_G2)_{\mathfrak{F}}$ given in Theorem 1.6 hold for f and A .

Then there exists a G -fiber-preserving map h , G -fiberwise homotopic to f with $A \subset \text{Fix}(h) \subset p^{-1}(p(A))$ and $\text{Fix}(\bar{h}) \cap (B - p(A))$ is a finite set.

Proof of Lemma 3.5: $p(A)$ is a closed G -subset of B then the G -fiber-preserving map $H_A : A \times I \rightarrow X$ given by $(C_G1)_{\mathfrak{F}}$ induces a G -map $\bar{H}_A : p(A) \times I \rightarrow B$ such that $\bar{H}_A(\bullet, 0) = \bar{f}$ and $\bar{H}_A(\bullet, 1) = i_{p(A)} : p(A) \hookrightarrow B$ the inclusion map.

Observe that we have almost the same conditions that we had in Theorem 1.5 except for (C_G2) . In this situation, suppose we have a G -map $\bar{H}_{i-1,A} : (p(A) \cup B_{i-1}) \times I \rightarrow B$. As commented in Lemma 2.5, it is possible to extend $\bar{H}_{i-1,A}$ to a G -map $\bar{H}_{i,1} : (B_i \cup p(A)) \times I \rightarrow B$ relative to $p(A) \cup B_{i-1}$.

Since WK_i is a finite group, $B_i^{K_i}$ is a WK_i -polyhedron such that $B_{i-1}^{K_i}$ is a WK_i -subpolyhedron of $B_i^{K_i}$ and $St(p(A_i^{K_i}))$ is neighborhood by-passed in $B_i^{K_i}$. Let V be a G -invariant neighborhood retract of $St(p(A_i)) \cup B_{i-1}$. It follows from Lemma 3.1 of [12] and Lemma 2.1 that there exists of a WK_i -homotopy $\bar{H}_i : B_i^{K_i} \times I \rightarrow B_i^{K_i}$ from $\bar{H}_{i,1}^{K_i}(\bullet, 1)$ to $\bar{h} = \bar{H}_i(\bullet, 1)$ such that:

1. $p(A)_i^{K_i} \subset \text{Fix}(\bar{h})$;
2. \bar{h} has a finite number of fixed points in $B_i^{K_i} - V^{K_i}$;
3. given a WK_i -fixed point class F of \bar{h} such that $F \cap p(A)_i^{K_i} = \emptyset$ then $F = WK_i\{x\}$, where $x \in B_i^{K_i} - V^{K_i}$ and F is an essential WK_i -fixed point class of \bar{h} .

Then, the G -map given by:

$$\bar{h}_t(x) = \begin{cases} g\bar{H}_i(g^{-1}x, t), & \text{for } x \in X - A, \text{ where } G_x = gWK_i g^{-1}; \\ \bar{h}(x), & \text{for } x \in V. \end{cases}$$

extends a WK_i -homotopy to a G -homotopy $\bar{H}_i : (B_i \cup p(A)) \times I \rightarrow B$ relative to V and such that

$$\text{Fix}(\bar{H}_i(\bullet, 1)) = p(A) \cup \left(\bigcup_{j \in T, j \leq i} (G\{b_{j,1}\} \cup \dots \cup G\{b_{j,m_j}\}) \right)$$

and $WK_i\{b_{i,l}\}$ is a essential WK_i -fixed point class of $\bar{H}_i^{K_i}(\bullet, 1)$, for $1 \leq l \leq m_i$.

Observe that if $p^K(F) = WK\{b_{i,l}\}$ for an essential WK -fixed point class F of f^K where $(K) \in \text{Iso}(X)$, then we have a path $\bar{\alpha}$ such that:

$$\{\bar{\alpha}\} \sim \{\bar{f}^K \circ \bar{\alpha}\} * \{\bar{H}_A^K(\bar{\alpha}(1), t)\} \sim \{\bar{H}^K(\bar{\alpha}(t), t)\}.$$

Hence, $\bar{\alpha}(1) = gb_{i,l}$, for some $g \in WK$ and $\bar{\alpha}(1) \in p^K(A)$. However, this cannot occur because $b_{i,l} \notin p^K(A)$ and $p^K(A)$ is WK -invariant. By induction we extend the G -map $\bar{H}_A : p(A) \times I \rightarrow B$ to a G -homotopy $\bar{H} : B \times I \rightarrow B$ with the properties above.

Note that $H' : X \times I \rightarrow B$ defined by $H'(x, t) = \bar{H}(p(x), t)$ is such that

$$H'(x, 0) = \bar{H}(p(x), 0) = \bar{f} \circ p(x) = p \circ f(x).$$

Therefore, the lift of H' is a fiber-preserving G -homotopy $H_1 : X \times I \rightarrow X$ such that $f(x) = H_1(x, 0)$ and $h_1(x) = H_1(x, 1)$. Thus,

$$\text{Fix}(h_1) \subset p^{-1}(\text{Fix}(\bar{h})) = p^{-1}(p(A) \cup G\{b_1\} \cup \dots \cup G\{b_l\}).$$

For each G -orbit $G\{b_j\}$ take the restriction h_{1,b_j} of h_1 for $Gp^{-1}(b_j) = p^{-1}(G\{b_j\})$, so $h_{1,b_j} : Gp^{-1}(b_j) \rightarrow Gp^{-1}(b_j)$ has no essential fixed point classes. In fact, suppose that h_{1,b_j}^K has an essential WK -fixed point class F . Then, given $x \in F$ we have $WK\{x\}$ lying inside an essential WK -fixed point class of h_1^K . Thus, there exists a WK -fixed point class Q of h_1^K which contains $WK\{x\}$. But, h_1^K is fiber-preserving WK -homotopic to f^K , so, there exists an essential WK -fixed point class D of f^K H_1^K -related to Q . Note that $p^K(D)$ cannot be \bar{H}^K -related to $WK\{b_j\}$. Consequently, h_{1,b_j} is fiber-preserving G -homotopic to $h_{2,b_j} : Gp^{-1}(b_j) \rightarrow Gp^{-1}(b_j)$ fixed point free.

Consider the G -map

$$\tilde{H}_2 : (X \times \{0\}) \cup (p^{-1}(\text{Fix}(\bar{h})) \times I) \rightarrow X$$

defined by:

$$\tilde{H}_2(x, t) = \begin{cases} h_1(x) & \text{if } t = 0 \text{ or if } x \in p^{-1}(p(A)); \\ H_{2,b_j}(x, t) & \text{if } x \in p^{-1}(G\{b_j\}). \end{cases}$$

With Proposition 3.4 we extend \tilde{H}_2 to a fiber-preserving G -homotopy $H_2 : X \times I \rightarrow X$ and $h_2 = H_2(\bullet, 1)$ is such that $\bar{H}_2(\bullet, 1) = \bar{h}$. By $(C_G 1)_{\mathfrak{F}}$, $h|_2$ is fiber-preserving G -homotopic to i_A . Let \tilde{H}_A such that $h_2|_A = \tilde{H}_A(\bullet, 0)$ and $i_A = \tilde{H}_A(\bullet, 1)$. Define $\tilde{H} : (X \times \{0\}) \cup ((A \cup p^{-1}(G\{b_1, \dots, b_r\})) \times I) \rightarrow X$ given by:

$$\tilde{H}(x, t) = \begin{cases} h_2(x), & \text{if } t = 0; \\ h_{2,b_j}(x), & \text{if } x \in Gp^{-1}(b_j); \\ \tilde{H}_A(x, t), & \text{if } x \in A. \end{cases}$$

Applying Proposition 3.4 again we extend \tilde{H} to a fiber-preserving G -homotopy $H : X \times I \rightarrow X$ such that $A \subset \text{Fix}(h) \subset p^{-1}(p(A))$ and $\text{Fix}(\bar{h}) \cap (B - p(A))$ is a finite set.

□

Lemma 3.6. Let $(\mathfrak{F}, \mathfrak{F}_0) = ((X, A), p, B, (Y, Y_0))$ be a G -fiber bundle pair, where X , B and Y are compact and smooth G -manifolds, B retracts equivariantly to a point $b_0 \in B$ and $\dim(Y^K) \geq 3$ and $\dim(Y^K) - \dim(Y^K - Y_K) \geq 2$, for all $(K) \in \text{Iso}(Y)$. Let Y_0 be a closed and locally contractible G -subset of Y such that Y_0^K is by-passed in Y^K , for all $(K) \in \text{Iso}(Y)$, A be a nonempty, closed, locally contractible G -subset of X and $f : X \rightarrow X$ be a G -map such that $p \circ f = p$, $A \subset \text{Fix}(f)$, A^K intersects every essential WK -fixed point class of $f_{b_0}^K : WK(p^K)^{-1}(\{b_0\}) \rightarrow WK(p^K)^{-1}(\{b_0\})$, for all $(K) \in \text{Iso}(X)$.

Then for every closed G -invariant subset Z of A that intersects every component of A and (A, Z) is G -fiber bundle pair of \mathfrak{F}_0 there exists a fiber-preserving G -map h , G -fiberwise homotopic to f with $\text{Fix}(h) = Z$.

Proof of Lemma 3.6: (X, p, B) is G -equivalent to a trivial G -fibration $(B \times Y, \pi, B)$, where π is a projection in B . So, there exists a G -homeomorphism $\Phi : B \times Y \rightarrow X$ such that $\Phi(B \times Y_0) = A$ and $p \circ \Phi = \pi$. Define $f^* = \Phi^{-1} \circ f \circ \Phi : B \times Y \rightarrow B \times Y$ and note that:

$$\pi \circ f^* = \underbrace{(p \circ \Phi) \circ (\Phi^{-1} \circ f \circ \Phi)}_{=p} = \underbrace{p \circ f \circ \Phi}_{=p} = p \circ \Phi = \pi.$$

Therefore, $f^*(b, y) = (b, f_b^*(y))$ and $gf^*(b, y) = (gb, f_{gb}^*(gy))$, for all $g \in G$.

B retracts equivariantly to b_0 , so, there exists a G -homotopy $D : B \times I \rightarrow B$ such that for each $b \in B$ we have $D(b, 0) = b$ and $D(b, 1) = b_0$. Then, define $U^* : B \times Y \times I \rightarrow B \times Y$ given by:

$$U^*(b, y, t) = \begin{cases} (b, f_{D(b, 2t)}^*(y)), & \text{if } 0 \leq t \leq \frac{1}{2} \\ (b, f_{D(b_0, 2-2t)}^*(y)), & \text{if } \frac{1}{2} \leq t \leq 1. \end{cases}$$

Note that $U^*(b, y, 0) = (b, f_b^*(y)) = f^*(b, y)$ and $U^*(b, y, 1) = (b, f_{b_0}^*(y))$. Then, f^* is G -homotopic to $id \times f_{b_0}^*$. Since $\Phi(B \times Y_0) = A \subset \text{Fix}(f)$ we have, for each $(b, y) \in B \times Y_0$:

$$f^*(b, y) = \Phi^{-1} \circ f \circ \underbrace{\Phi(b, y)}_{\substack{\in A \\ =\Phi(b, y)}} = \Phi^{-1} \circ \Phi(b, y) = (b, y).$$

Then, $B \times Y_0 \subset \text{Fix}(f^*)$ and $Y_0 \subset \text{Fix}(f_b^*)$ because $(b, f_b^*(y)) = f^*(b, y) = (b, y)$.

By hypothesis, A intersects each essential WK -fixed point class of $f_{b_0}^K : WK(p^K)^{-1}(b_0) \rightarrow WK(p^K)^{-1}(b_0)$. So, $A \cap WK(p^K)^{-1}(b_0)$ intersects each essential WK -fixed point class of $f_{b_0}^K$. So, Y_0 intersects each essential WK -fixed point class of $(f_{b_0}^*)^K$ because:

$$(\Phi^{-1})^K(A^K \cap WK(p^K)^{-1}(b_0)) = WK\{b_0\} \times Y_0^K.$$

The G -fiber bundle pair $((A, Z), p, B, (Y_0, \Omega))$ is such that Ω intersects every component of Y_0 because Z intersects every component of A . Therefore, $(C_G 1)_{\mathfrak{F}}$ and $(C_G 2)_{\mathfrak{F}}$ hold for Y_0 and $f_{b_0}^*$. By Theorem 2.2 there exists a homotopy $V^* : Y \times I \rightarrow Y$ such that $f_{b_0}^* = V^*(\bullet, 0)$, $V^*(\bullet, 1) = g_{b_0}^* : Y \rightarrow Y$ and $\text{Fix}(g_{b_0}^*) = \Omega$. Define a fiber-preserving G -homotopy $H^* : B \times Y \times I \rightarrow B \times Y$ given by:

$$H^*(b, y, t) = \begin{cases} U^*(b, y, 2t), & \text{if } 0 \leq t \leq \frac{1}{2} \\ (b, V^*(y, 2t - 1)), & \text{if } \frac{1}{2} \leq t \leq 1. \end{cases}$$

Therefore, f^* is G -homotopic to $\text{id} \times g_{b_0}^*$ and $\text{Fix}(\text{id} \times g_{b_0}^*) = B \times \Omega$. Then, $H : X \times I \rightarrow X$ given by $H(e, t) = \Phi \circ H^*(\Phi^{-1}(e), t)$ is a G -homotopy such that $H(x, 0) = f(x)$ and $\text{Fix}(H(\bullet, 1)) = \Phi(B \times \Omega) = Z$.

□

Proof of Theorem 1.6: With Lemma 3.5 we assume that:

1. $A \subset \text{Fix}(f) \subset p^{-1}(p(A))$;
2. $F = \text{Fix}(\bar{f}) \cap (B - p(A))$ is a finite set.

Let $f_j = f|_{p^{-1}(p(A)_j)} : p^{-1}(p(A)_j) \rightarrow p^{-1}(p(A)_j)$ a restriction of f , so $p \circ f_j = p$. Using $X = p^{-1}(p(A)_j)$, $A = A \cap p^{-1}(p(A)_j)$, $B = p(A)_j$, $b_0 = b_j$, $Y_0 = Y_j$ and $f = f_j$ the hypotheses of Lemma 3.6 are satisfied and there exists a fiber-preserving G -homotopy $H_j : p^{-1}(p(A)_j) \times I \rightarrow p^{-1}(p(A)_j)$ from f_j to $h_j = H_j(\bullet, 1)$ such that $\text{Fix}(h_j) = Z_j$.

Define $\tilde{H}_2 : (X \times \{0\}) \cup (p^{-1}(F \cup p(A)) \times I) \rightarrow X$ by:

$$\tilde{H}_2(x, t) = \begin{cases} f(x), & \text{if } t = 0 \text{ or } p(x) \in F \\ H_j(x, t), & \text{if } p(x) \in p(A)_j. \end{cases}$$

With Proposition 3.4 there is a fiber-preserving G -homotopy $H : X \times I \rightarrow X$ such that $p(H(x, t)) = \bar{f} \circ p(x)$. Therefore, $h = H(\bullet, 1) : X \rightarrow X$ is such that $\text{Fix}(h) = Z$ and h is fiber-preserving G -homotopic to f .

□

Corollary 3.7. Let $\mathfrak{F} = (X, p, B, Y)$ be a G -fiber bundle where X , B and Y are compact and smooth G -manifolds, $\dim(B^K) \geq 3$, $\dim(B^K) - \dim(B^K - B_K) \geq 2$, for all $(K) \in \text{Iso}(B)$, $\dim(Y^K) \geq 3$, $\dim(Y^K) - \dim(Y^K - Y_K) \geq 2$, for all $(K) \in \text{Iso}(Y)$.

Let A be a nonempty, closed, locally contractible G -subset of X such that (X, A) is G -fiber bundle pair with respect to the fiber bundle \mathfrak{F} , $p(A)$ be a closed G -subset of B such that

each component $p(A)_j$ of $p(A)$ is equivariantly contractible and $p^K(A^K)$ is by-passed in B^K , for all $(K) \in \text{Iso}(B)$. Let Y_j be a subbundle fiber of A such that Y_j is a closed and locally contractible G -subset of Y and Y_j^K is by-passed in Y^K , for all $(K) \in \text{Iso}(Y)$, and $f : X \rightarrow X$ be a G -fiber-preserving map such that A^K intersects every essential WK -fixed point class of $f_{b_j}^K : WK(p^K)^{-1}(\{b_j\}) \rightarrow WK(p^K)^{-1}(\{b_j\})$ for at least one b_j in each component $p^K(A^K)_j$, for all $(K) \in \text{Iso}(X)$.

Then there exists a G -fiber-preserving map h , G -fiberwise homotopic to f with $\text{Fix}(h) = A$ if, and only if, the following conditions holds for f and A :

- $(C_G1)_{\mathfrak{F}}$ there exists a G -fiber-homotopy $H_A : A \times I \rightarrow X$ from $f|_A$ to the inclusion $i : A \hookrightarrow X$;
- $(C_G2)_{\mathfrak{F}}$ for every WK -essential fixed point class F of $f^K : X^K \rightarrow X^K$ there exists a path $\alpha : I \rightarrow X^K$ with $\alpha(0) \in F$, $\alpha(1) \in A^K$, and $\{\alpha(t)\} \sim \{f^K \circ \alpha(t)\} * \{H_A^K(\alpha(1), t)\}$.

Proof of Corollary 3.7: If the conditions hold then we apply Theorem 1.6 for $Z = A$. If there exists h then by Lemma 3.2 $(C_G1)_{\mathfrak{F}}$ and $(C_G2)_{\mathfrak{F}}$ hold.

□

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